JACAL

Symbolic Mathematics System
Version 2a1, August 2026

Aubrey Jaffer

This manual is for JACAL (version 2a1, August 2026), an interactive symbolic mathematics system.

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1 Overview

JACAL is an interactive symbolic mathematics program. JACAL can manipulate and simplify equations, scalars, vectors, matrices, and tensors of (single and multiple valued) algebraic expressions containing numbers, variables, radicals, and algebraic and transcendental functions.

JACAL 2a1 was released August 2026. Current information about JACAL can be found on JACAL's *WWW* home page:

`https://www.gnu.org/software/jacal/`

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For a list of the features that have changed since the last JACAL release, see the file `ANNOUNCE`. For a list of the features that have changed over time, see the file `ChangeLog`.

1.1 Authors and Bibliography

Michael Thomas
: Polynomial Factoring.

Jerry D. Hedden
: Tensors.

Aubrey Jaffer
: Most of JACAL

The maintainer can be reached as '`agj@alum.mit.edu`'.

Project History

In December of 1987, in order to facilitate the design of constant impedance electrical filters (diplexers), I (Aubrey Jaffer) wrote a symbolic circuit analysis program. It was written in LISP and implemented canonical rational expressions (ratio of multivariate polynomials) as its representation of the (small signal) Laplace Transform of currents, voltages, and impedances. After doing some reading about symbolic manipulation, I became fascinated with the problem of canonical forms. That interest produced JACAL, an interactive symbolic mathematics program similar to Maxima.

I initially used JACAL for electronics modeling, handling equations complicated enough that error-free pencil and paper manipulations were difficult. As JACAL's capabilities grew, more applications were found.

Around 1993-1995 Mike Thomas added univariate and multivariate polynomial factorization. In 1993-1997, Jerry D. Hedden added tensor manipulations to JACAL.

Having retired, in 2024 I finally implemented indefinite integration (actually anti-differentiation) of rational functions incorporating radicals.

Spurred by my mathematical physics work in fluid-mechanics, I implemented logarithms, exponentials, Lambert-W (the inverse of `x*exp(x)`), and trigonometric functions as the solutions to ordinary differential equations (ODE).

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SLIB; The Portable Scheme Library (`../slib_toc`)

1.2 Installation

The JACAL program is written in the Algorithmic Language *Scheme*. So you must obtain and install a Scheme implementation in order to run it. The installation procedures given here use the SCM Scheme implementation. If your system has a Scheme (or Guile) implementation installed, then the '`scm`' steps are unnecessary.

JACAL also requires the SLIB Portable Scheme library which is available from `https://www.gnu.org/software/slib`.

`Unix` [System]

`GNU/Linux` [System]

```
wget https://ftp.gnu.org/gnu/scm/scm-5f5.tar.gz
wget https://ftp.gnu.org/gnu/slib/slib-3c2.tar.gz
wget https://ftp.gnu.org/gnu/jacal/jacal-2a1.tar.gz
tar -xzf scm-5f5.tar.gz
tar -xzf slib-3c2.tar.gz
tar -xzf jacal-2a1.tar.gz
(cd slib-3c2; ./configure --prefix=/usr/local/; make install)
(cd scm-5f5; ./configure --prefix=/usr/local/; make scm; make install)
(cd jacal-2a1; ./configure --prefix=/usr/local/; make install)
# rm scm-5f5.tar.gz slib-3c2.tar.gz jacal-2a1.tar.gz
```

The command '`jacal`' will start an interactive session.

`x86_64` *GNU/Linux with Redhat Package Manager* (*rpm*) [System]

```
wget http://groups.csail.mit.edu/mac/ftpdir/scm/scm-5f5-1.x86_64.rpm
wget http://groups.csail.mit.edu/mac/ftpdir/scm/slib-3c2-1.noarch.rpm
wget http://groups.csail.mit.edu/mac/ftpdir/scm/jacal-2a1-1.noarch.rpm
rpm -U scm-5f5-1.x86_64.rpm slib-3c2-1.noarch.rpm jacal-2a1-1.noarch.rpm
rm scm-5f5-1.x86_64.rpm slib-3c2-1.noarch.rpm jacal-2a1-1.noarch.rpm
```

The command '`jacal`' will start an interactive session using ELK, Gambit, Gauche, Guile, Larceny, MIT-Scheme, MzScheme, Scheme48, SCM, or SISC. Type '`jacal --help`' for instructions.

`x86` *Microsoft* [System]

Download and run `http://groups.csail.mit.edu/mac/ftpdir/scm/slib-3c2-1.exe`,
`http://groups.csail.mit.edu/mac/ftpdir/scm/scm-5f5-1.exe`, and
`http://groups.csail.mit.edu/mac/ftpdir/scm/jacal-2a1-1.exe`.

Manifest

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: details the LACK OF WARRANTY for Jacal and the conditions for distributing Jacal.

HELP
: is a short introduction to using Jacal.

ChangeLog
: documents changes to Jacal.

jacal is a unix (sh) script to start an interactive jacal session.

rw.math is a batch file of Robertson-Walker model of General Relativity. "batch(rw.math);" to execute its commands in Jacal.

test.math is a batch file which tests Jacal.

jacal.texi is documentation on how to use jacal in TeXinfo format.

DOC has files telling about how jacal works.

- algdenom gives an algorithm for clearing radicals and other algebraic field extensions from denominators.
- grammar explains how to create new grammars.
- lambda explains mid-level data formats. From a Dr. Dobbs article.
- ratint.pdf article explaining jacal's integration algorithm and transcendental function simplification.

math.scm is the file you load into scheme in order to run jacal.

toploads.scm
contains comments describing the rest of the files.

modeinit.scm
has initializations for modes in Jacal.

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2 Introduction

2.1 Running Jacal

If you successfully executed one of the installations of the previous section, then typing '`jacal`' or clicking an icon will begin an interactive session.

To manually start jacal, start your Scheme implementation with SLIB. This may involve setting up that implementation's initialization file or LOADing a '`.init`' file from the `slib` directory. Then type:

```
(slib:load "/usr/local/lib/jacal/math")
```

where `/usr/local/lib/jacal/` is a path to the JACAL directory. JACAL should then print:

```
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JACAL comes with ABSOLUTELY NO WARRANTY; for details type `(terms)'.
This is free software, and you are welcome to redistribute it
under certain conditions; type `(terms)' for details.
;;; Type (math); to begin from Scheme session.
```

Do as it says:

```
(math)
⇒
type qed; to return to scheme, type help; for help.
(%0000)
```

And you are ready to try the commands described in the rest of the manual.

Recovery from Errors

JACAL tries to catch any errors it encounters and print an informative message before returning to the prompt. If there are unmatched parentheses or missing delimiters in your typed command, type the closing delimiters, digits for "expression missing", and semicolon (followed by newline) until it returns to the prompt:

```
(%0000) ({adf;
          ^ "mismatched delimiter" #\; expecting #\}
;
^ "mismatched delimiter" #\; expecting #\)
;
%0000: adf
(%0001)
```

If that is unsuccessful, typing '`CTRL-C`' will likely return you to the underlying Scheme session.

As JACAL is a complicated program there are bugs which will occasionally cause the program to stop with some sort of error reported by the underlying Scheme system. In interactive implementations (such as SCM) you can usually continue your session by typing `(math)`. The expression which was input to JACAL just before the error will be lost but you should be able to otherwise continue with your session.

If you define identifiers which have already been used in formulas, the earlier uses are not updated. You can restore a name to its unassigned status by defining it to itself:

```
(%0001) a: 5/7;
a: 5/7
(%0001) a+3;
       26
%0001: --
       7
(%0002) a:a;
a: a
(%0002) a+3;
%0002: 3 + a
```

If the underlying scheme implementation supports a `(restart)` command, then typing `restart();` in JACAL will re-initiate the session, forgetting all the commands and formulas you entered previously:

```
(%0003) restart();
..............
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;;; Type (math); to begin from Scheme session.
Type qed; to return to Scheme, type help; for help.
(%0000)
```

Stopping Jacal

The command `quit();` will end your JACAL session.

With non-interactive Scheme implementations the JACAL command `qed;` or typing the end-of-file character (`C-z` on MS-DOS and VMS, `C-d` on others) will end your JACAL session.

The command `qed;` will return to the interactive Scheme session. Typing `(math)` will return to the JACAL session.

From the interactive Scheme session `(exit)` or possibly an end-of-file character will terminate the session.

2.2 Canonical

The canonical aspect of JACAL is that all equations and expressions involving variables undergo nomalization such that two formulas which are equivalent have the same representation. This representation is affected by variable ordering; substituting one variable name for another looks different but expresses the same mathematical constraint.

As with all symbolic algebra systems, roots of unity (1) are not uniquely normalizable. JACAL's guarantees apply only to expressions involving variables.

JACAL prioritizes simplicity of expressions over full canonical reductions. For instance, all trigonometric functions can be expressed as (complex number) exponentials and

logarithms; but it is difficult to understand that representation. Instead, trigonometric expressions are represented in terms of tangent (tan) and arctangent (atan).

A transcendental function composed with its inverse cancels each other if the argument to the inverse function involves at least one variable. `exp(integer*log(argument))` reduces to `argument^(integer)` if *argument* involves at least one variable. `exp(log(argument)/integer)` does not reduce if *integer* is greater than one. The `^` function reduces this case to a radical:

```
(%0000) %expt(a+x,1/5);
           log(a + x)
%0000: exp(----------)
                5
(%0001) (a+x)^(1/5);
              1/5
%0001: (a + x)
```

Trignometric functions have similar reductions:

```
(%0002) sin(asin(x));
%0002: x
(%0003) asin(sin(x));
%0003: x
(%0004) sin(3*asin(x));
                    3
%0004: - 3 x + 4 x
(%0005) tan(3*atan(x));
                 3
       - 3 x + x
%0005: ----------
                 2
       -1 + 3 x
(%0006) atan(3*tan(x));
%0006: atan(3 tan(x))
(%0007) tan(6*atan(x));
                    3      5
         6 x - 20 x  + 6 x
%0007: ----------------------
                 2       4    6
       1 - 15 x  + 15 x  - x
```

The Lambert-W function (`%W`) is the inverse of `z*exp(z)`. Currently the reduction works in one direction; the other will eventually be supported:

```
(%0008) %W(x*exp(x));
%0008: x
(%0009) %W(x)*exp(%W(x));
%0009: %W(x) exp(%W(x))
```

Because transcendental functions are defined by their differential equations, JACAL can produce their derivatives:

```
(%0015) diff(%W(x),x);
          %W(x)
%0015: -----------
       x + x %W(x)
(%0016) diff(tan(x),x);
                    2
%0016: 1 + (tan(x))
(%0017) diff(sin(x),x);
                1
%0017: ------------------
                     2 1/2
       (1 + (tan(x)) )
(%0018) diff(cos(x),x);
              tan(x)
%0018: --------------------
                       2 1/2
       - (1 + (tan(x)) )
(%0019) diff(%expt(x,1/5),x);
           log(x)
       exp(------)
             5
%0019: -----------
           5 x
(%0020) diff(x^(1/5),x);
         1
%0020: ------
          4/5
       5 x
```

The usual algebraic reductions also work:

```
(%0024) sin(x/a)^2+cos(x/a)^2;
%0024: 1
```

JACAL provides some functions which it does not canonicalize: `imagpart`, `realpart`, `abs`, and `cabs`.

2.3 Release Notes

JACAL-2a1 is a major release incorporating transcendental functions.

Do not raise transcendental functions to powers like `sin^2(x)`. Instead, use `sin(x)^2`.

Trigonometric sum and difference of angle formulas do not reduce.

2.4 Conventions

Things that are labeled as Operators can occur in expressions output by Jacal. Things that are labeled as Commands act upon their arguments and do not generally occur in expressions output by Jacal. Things that are labeled as flags are `set` to control aspects of the Jacal environment.

The examples throughout this text were produced using the SCM Scheme implementation.

Jacal has several input grammars it understands. The `standard` input grammar (which is the same as the `std` and `disp2d` input grammars) is used in this manual. It is very similar to `Maxima` input grammar and the Algol family of computer languages.

Identifier names are case sensitive.

3 Algebra

3.1 Algebraic Operators

`+` *augend addend* [Operator]

Addition of scalar quantities or componentwise addition of bunches is accomplished by means of the infix operator `+`. For example,

```
(%0000) a:[[1, 3, 5], [2, 4, 7]];
   [ 1  3  5 ]
a: [         ]
   [ 2  4  7 ]
(%0000) b:[2, 4];
b: [2, 4]
(%0000) a + b;
       [ 3  5  7  ]
%0000: [          ]
       [ 6  8  11 ]
(%0001) 3 + 2;
%0001: 5
(%0002) c + b;
%0002: [2 + c, 4 + c]
```

`-` *minuend subtrahend* [Operator]

`-` *subtrahend* [Operator]

The symbol `-` is used to denote either the binary infix operator subtraction or the unary minus.

```
(%0004) -[1,2,3];
%0004: [-1, -2, -3]
(%0005) 3-7;
%0005: -4
```

`+/-` *minuend subtrahend* [Operator]

`-/+` *minuend subtrahend* [Operator]

`+/-` *augend* [Operator]

`-/+` *augend* [Operator]

Jacal allows the use of `+/-` and `-/+` as ambiguous signs (unary plus-or-minus, unary minus-or-plus) and as ambiguous infix operators (binary plus-or-minus, binary minus-or-plus). The value `+/-` is also represented by the constant `%sqrt1`, while `-/+` is represented by `-%sqrt1`.

```
(%0006) u:+/-3;
u: 3 %sqrt1
(%0006) u^2;
%0006: 9
(%0007) +/-(u);
%0007: 3
(%0008) u-/+3;
%0008: 0
```

`*` *multiplicand1 multiplicand2* [Operator]

Multiplication of scalar expressions such as numbers, polynomials, rational functions and algebraic functions is denoted by the infix operator `*`. For example,

```
(%0000) (2 + 3 * a) * 4 * a * b^2;
                     2   2
%0000: (8 a + 12 a ) b
```

One can also use `*` as an infix operator on bunches. In that case, it operates componentwise, in an appropriate sense. If the bunches are square matrices, the operator `*` multiplies corresponding entries of the two factors. It does not perform matrix multiplication. To multiply matrices one instead uses the operator `.` (i.e., a period). More generally, any binary scalar operator other than `^` can be used on bunches and acts componentwise.

`/` *dividend divisor* [Operator]

The symbol for division in Jacal is `/`. For example, the value returned by `6 / 2` is `3`.

```
(%0001) (x^2 - y^2) / (x - y);
%0001: x + y
```

Note: using `divide` to divide a polynomial by an integer does not work.

`^` *expression exponent* [Operator]

The infix operator `^` is used for exponentiation of scalar quantitites or for componentwise exponentiation of bunches. For example, `2^5` returns 32. Unlike the other scalar infix operators, one cannot use `^` for component-wise operations on bunches. Furthermore, one should not try to use `^` to raise a square matrix to a power. Instead, one should use `^^`.

```
(%0002) (1+x)^4;
                        2      3    4
%0002: 1 + 4 x + 6 x  + 4 x  + x
```

`=` *expression1 expression2* [Operator]

In Jacal, the equals sign `=` is *not* used for conditionals and it is *not* used for assignments. To assign a value to an identifier, use either `:` or `:=`. The operator `=` merely returns a value of the form `0 =` ***`expression`***. The value returned by `a = b`, for example is `0 = a - b`.

```
(%0003) 1=2;

;;; eliminate singular-reduction [-1]
```

`||` *Z1 Z2* [Operator]

The infix operator `||` is from electrical engineering and represents the effective impedance of the parallel connection of components of impedances *Z1* and *Z2*:

```
(%0003) Z1 || Z2;
        Z1 Z2
%0003: -------
       Z1 + Z2
```

3.2 Algebraic Commands

`eliminate` [*eqn_1 eqn_2 ...*] [*var_1 var_2 ...*] [Command]

Here *eqn_i* is an equation for $i = 1 \ldots n$ and where *var_j* is a variable for $j = 1 \ldots m$. `eliminate` returns a list of equations obtained by eliminating the variables *var_1*, ..., *var_m* from the equations *eqn_1*, ..., *eqn_n*.

```
(%0004) eliminate([x^2+y=0,x^3+y=0],[x]);
                    2
%0004: 0 = y + y
(%0005) eliminate([x+y+z=3,x^2+y^2+z^2=3,x^3+y^3+z^3=3],[x,y]);
%0005: 0 = -1 + z
```

`suchthat` *var eqn* [Command]

The equation *eqn* must contain an occurrence of variable *var*. `suchthat` returns an expression for all complex values of *var* satisfying *eqn*. `suchthat` is useful for extracting an expression from an equation.

```
(%0000) a*x+b*y+c = 0;
%0000: 0 = c + a x + b y
(%0001) suchthat(x,%0000);
        - c - b y
%0001: ---------
             a
```

`suchthat` *var exp* [Command]

If an expression rather than an equation is given to `suchthat`, it is as though the equation *exp*`=0` was given.

```
(%0002) suchthat(x, a*x+b*y+c);
        - c - b y
%0002: ---------
             a
```

`|` *var exp_or_eqn* [Operator]

An alternative infix notation is also available for `suchthat`.

When used in combination with the '`{ }`' notation for `or`, the set notation used by some textbooks results.

If *var* in *eqn* has multiple roots, a named *ring extension* will be introduced to represent any one of those roots. When multiple values are returned, the result (in `disp2d` and `standard` grammars) is wrapped with '`{ }`'.

```
(%0003) x | a*x^2 + b*x + c;
                                2
%ext1: c + b %ext1 + a %ext1
(%0003) %ext1^2;
       - c - b %ext1
%0003: -------------
              a
```

`definition` *symbol* [Command]

Returns the rule or expression defining *symbol*.

```
(%0000) {x | a*x^2 + b*x + c};
                                2
%ext1: c + b %ext1 + a %ext1
(%0000) definition(%ext1);
                                     2
%0000: 0 = c + b %ext1 + a %ext1
(%0001) definition(sqrt);
                      2/2
%0001: 0 = - @1 + @1
(%0002) definition(sqrt(x));
                    2/2
%0002: 0 = - x + x
(%0003) definition(tan);
                             2
%0003: 0 = - @1' + (1 + @1 ) @2'
(%0004) definition(tan(x));
                                              2
%0004: 0 = - (tan(x))' + (1 + (tan(x)) ) x'
(%0005) definition(%tanP);
%0005: %tanP          + @2 %tanQ
            [-1 + @1]           [-1 + @1]
(%0006) definition(%tanPQ);
       %tanP
            [@1]
%0006: ---------
       %tanQ
            [@1]
```

`or` *expr_1 ...* [Command]

`or` *eqn_1 ...* [Command]

The function `or` takes as inputs one or more equations or values. If the inputs are equations, then `or` returns an equation which is equivalent to the assertion that at least one of the input equations holds. If the inputs to `or` are values instead of two equations, then the function `or` returns a multiple value. If the inputs to `or` consist of both equations and values, then `or` will return the multiple values.

```
(%0000) or(x=2,y=3);
%0000: 0 = 6 - 3 x + (-2 + x) y
(%0001) or(2,3);
                              2
%0001: {@ | 0 = 6 - 5 @ + @ }
(%0002) or(2,3)^2;
                                2
%0002: {@ | 0 = 36 - 13 @ + @ }
(%0003) factor %0001;
%0003: 3 2
(%0004) factor %0002;
%0004: 9 4
(%0005) or(x=2,17);
%0005: 17
```

'`{eqn, ... }`' can be used as an alternate syntax for `or`:

```
(%0012) : {+1, -1};

                         2
%0012: {:@ | 0 = -1 + :@ }
```

3.3 Rational Expression

`num` *expr* [Command]

The function `num` takes a rational expression as input and returns a numerator of the expression.

```
(%0013) num((x^2+y^2)/(x^2-y^2));
          2    2
%0013: - x  - y
(%0014) num(7/4);
%0014: 7
(%0015) num(7/(4/3));
%0015: 21
```

`denom` *rational-expression* [Operator]

The Jacal command `denom` is used to obtain the denominator of a rational expression.

```
(%0016) denom(4/5);
%0016: 5
(%0017) denom((x^2+y^2)/(x^2-y^2));
          2    2
%0017: - x  + y
```

`listofvars` *expr* [Command]

The command `listofvars` takes as input a rational expression and returns a list of the variables that occur in that expression.

```
(%0018) listofvars(x^2+y^3);
%0018: [x, y]
(%0019) listofvars((x^2+y^3)/(2*x^7+y*x+z));
%0019: [x, y, z]
```

`imagpart` *z* [Command]
Returns the coefficient of `%i` in expression *z*;

`realpart` *z* [Command]
Returns all but the coefficient of `%i` in expression *z*;

`abs` *z* [Command]
`cabs` *z* [Command]
`| z |`
Returns the square root of the sum of the squares of the `realpart` and the `imagpart` of *z*.

```
(%0020) abs(z);
%0020: |z|
(%0021) abs(-z);
%0021: |- z|
(%0022) abs(-3);
%0022: 3
(%0023) abs(1/%i);
%0023: 1
(%0024) realpart(1/%i);
%0024: 0
(%0025) imagpart(1/%i);
%0025: -1
(%0026) imagpart(3/%i);
%0026: -3
```

3.4 Polynomials

`degree` *poly var* [Operator]
Returns the degree of polynomial or equation *poly* in variable *var*.

`degree` *poly* [Operator]
Returns the total-degree, the degree of its highest degree monomial, of polynomial or equation *poly*.

```
(%0028) degree(a*x*x + b*y*x + c*y*y + d*x + e*y + f, y);
%0028: 2
(%0029) degree(a*x*x + b*y*x + c*y*y + d*x + e*y + f);
%0029: 3
```

Note that all roots and radicals count as degree 1.

`coeff` *poly var* [Operator]
`coeff` *poly var deg* [Operator]

`coeffs` *poly var* [Operator]

The command `coeff` is used to determine the coefficient of a certain power of a variable in a given polynomial. Here *poly* is a polynomial and *var* is a variable. If the optional third argument is omitted, then Jacal returns the coefficient of the variable *var* in *poly*. Otherwise it returns the coefficient of *var^deg* in *poly*. The function `coeffs` returns a list of all of the coefficients. For example,

```
(%0009) coeff((x + 2)^4, x, 3);
%0009: 8
(%0010) (x + 2)^4;
                         2      3    4
%0010: 16 + 32 x + 24 x  + 8 x  + x
(%0011) coeff((x + 2)^4, x);
%0011: 32
(%0012) coeffs((x + 2)^4, x);
%0012: [16, 32, 24, 8, 1]
```

`poly` *var vect* [Operator]

`poly` *var coeff1 ...* [Operator]

The function `poly` provides an inverse to the function `coeffs`, allowing one to recover a polynomial from its vector or list of coefficients.

```
(%0013) poly(y, [16, 32, 24, 8, 1]);
                         2      3    4
%0013: 16 + 32 y + 24 y  + 8 y  + y
(%0014) poly(y, 16, 32, 24, 8, 1);
                         2      3    4
%0014: 16 + 32 y + 24 y  + 8 y  + y
```

`poly` *eqn* [Operator]

The function `poly` returns the expression equal to 0 in equation *eqn*. Be aware that the sign and scaling of the returned polynomial will not necessarily match those in the equation creating *eqn*.

```
(%0015) 2*a = 4*c;
%0015: 0 = - a + 2 c
(%0016) poly(%0015);
%0016: - a + 2 c
```

`content` *poly var* [Operator]

Returns a list of content and primitive part of a polynomial with respect to the variable. The content is the GCD of the coefficients of the polynomial in the variable. The primitive part is *poly* divided by the content.

```
content(2*x*y+4*x^2*y^2,y);
                        2
%0017: [2 x, y + 2 x y ]
```

`divide` *dividend divisor var* [Operator]
`divide` *dividend divisor* [Operator]

The command `divide` treats *divident* and *divisor* as polynomials in the variable *var* and returns a pair '`[quotient, remainder]`' such that `dividend = divisor * quotient + remainder`. If the third argument *var* is omitted Jacal will choose a variable on its own with respect to which it will do the division. In particular, of *dividend* and *divisor* are both numerical, one can safely omit the third argument.

```
(%0018) divide(x^2+y^2,x-7*y^2,x);
                2   2       4
%0018: [x + 7 y , y  + 49 y ]
(%0019) divide(-7,3);
%0019: [-2, -1]
(%0020) divide(x^2+y^2+z^2,x+y+z);
                          2             2
%0020: [- x - y + z, 2 x  + 2 x y + 2 y ]
(%0021) divide(x^2+y^2+z^2,x+y+z,y);
                          2             2
%0021: [- x + y - z, 2 x  + 2 x z + 2 z ]
(%0022) divide(x^2+y^2+z^2,x+y+z,z);
                          2             2
%0022: [- x - y + z, 2 x  + 2 x y + 2 y ]
```

`mod` *poly1 eqn var* [Command]
`mod` *poly1 poly2 var* [Command]
`mod` *poly1 poly2* [Command]

Returns *poly1* reduced with respect to *poly2* (or *eqn*) and *var*. If *poly2* is univariate, the third argument is not needed.

`mod` *poly1 n* [Command]

Returns *poly1* with all the coefficients taken modulo *n*.

`mod` *poly1* [Command]

Returns *poly1* with all the coefficients taken modulo the current modulus.

If the modulus (*n* or the current modulus) is negative, then the results use symmetric representation.

```
(%0023) x^4+4 mod 3;
             4
%0023: 1 + x
(%0024) x^4+4 mod x^2=2;
%0024: 8
(%0025) mod(x^3*a*7+x*8+34, -3);
                    3
%0025: 1 - x + a x
(%0026) mod(5,2);
%0026: 1
(%0027) mod(x^4+4,x^2=2,x);
%0027: 8
```

`gcd` *poly_1 poly_2* [Command]

The Jacal function `gcd` takes as arguments two polynomials with integer coefficients and returns a greatest common divisor of the two polynomials. This includes the case where the polynomials are integers.

```
(%0028) gcd(x^4-y^4,x^6+y^6);
        2    2
%0028: x  + y
(%0029) gcd(4,10);
%0029: 2
```

`discriminant` *poly var* [Command]

Here *poly* is a polynomial and *var* is a variable. This function returns the square of the product of the differences of the roots of the polynomial *poly* with respect to the variable *var*.

```
(%0030) discriminant(x^3 - 1, x);
%0030: -27
```

`resultant` *poly_1 poly_2 var* [Command]

The function `resultant` returns the resultant of the polynomials *poly_1* and *poly_2* with respect to the variable *var*.

```
(%0031) resultant(x^2 + a, x^3 + a, x);
        2    3
%0031: a  + a
```

`equatecoeffs` *z1 z2 var* [Command]

Returns the list of equations formed by equating each coefficient of variable *var*^n in *z1* to the corresponding coefficient of *var*^n in *z2*. *z1* and *z2* can be polynomials or ratios of polynomials.

`decompose` *poly_1 var* [Command]

Returns the polynomial decomposition of *poly_1* with respect to *var*. Note that `decompose` is not currently working.

3.5 Interpolation

`interp` *mat* [Command]

`interp` *vec1 vec2 . . .* [Command]

The only argument, *mat*, must be an array having at least one row of two expressions: [[*x1*,*y1*],[*x2*,*y2*],. . .]. It is an error if there are any duplicates in the first column of the second argument,

`interp` returns a polynomial function *poly*(*@1*) such that *mat*[1,2]=*poly*(*mat*[1,1]), *mat*[2,2]=*poly*(*mat*[2,1]), etc.

There is a variant of the `interp` command that takes multiple vector arguments instead of a matrix. These vectors represent points to be interpolated over. The same constraints apply as in the matrix version. All the variants of the interpolation procedure described later have both these forms.

```
(%0032) interp([[2, 3], [0, -1]]);
%0032: -1 + 2 @1
(%0033) interp([[2, 3], [1, z]]);
%0033: -3 + 3 @1 + (2 - @1) z
(%0034) interp([2, 3], [y, z]);
       - 3 @1 + 3 y + (-2 + @1) z
%0034: -------------------------
                 -2 + y
```

`interp.lagrange` *mat* [Command]
`interp.lagrange` *vec1 vec2 ...* [Command]
This is the same as the `interp` command.

`interp.newton` *mat* [Command]
`interp.newton` *vec1 vec2 ...* [Command]
This is similar to `interp` command with an added option of including derivative values when defining points. The same constraints apply as in `interp`. You can choose to specify some number of derivatives for each point. That number does not have to be the same for all points.

```
(%0035) interp.newton([-1, 0], [0, 1], [1, 0]);
             2
%0035: 1 - @1
(%0036) interp.newton([-1, 0], [0, 1, 0, 20], [1, 0]);
                2        4
%0036: 1 + 10 @1  - 11 @1
(%0037) interp.newton([-1, 0], [0, 1, 0, a], [1, 0]);
               4      2     4
       2 - 2 @1  + (@1  - @1 ) a
%0037: -------------------------
                   2
```

`interp.neville` *mat* [Command]
`interp.neville` *vec1 vec2 ...* [Command]
The same as `interp` in its functionality, but uses Neville form when constructing the polynomial.

3.6 Factoring

`factor` *int* [Command]
The Jacal command `factor` takes as input an integer and returns a list of the prime numbers that divide it, each occurring with the appropriate multiplicity in the list. If the number is negative, the list will begin with -1.

The results of the `factor` command are shown in a special *factored* format, which appears as the product of the factors.

```
(%0038) factor(120);
        3
%0038: 2  3 5
(%0039) factor(-120);
           3
%0039: -1 2  3 5
```

`factor` *polyratio* [Command]

Given a univariate ratio of polynomials *polyratio*, returns a matrix of factors and exponents.

As above, the results are shown in factored form.

```
(%0040) factor((14*x^4-10/68*x^-5)/(5*x^2+1));
                     9
          -5 + 476 x
%0040: ------------------
                     2   5
       2 17 (1 + 5 x ) x
(%0043) factor(x*y);
%0043: x y
(%0044) factor((x+a)*(y^4-z));
                      4
%0044: -1 (a + x) (- y  + z)
(%0045) factor((x+u*a^3)*(y^4-z));
             3           4
%0045: -1 (a  u + x) (- y  + z)
(%0046) factor((x+u*a^3)^2*(y^4-z)/((x+1)*(u^2-v^2)));
            4        3       2
        (- y  + z) (a  u + x)
%0046: -------------------------
       (- u + v) (u + v) (1 + x)
(%0047) factor(200*(-1*x+1+y)*(u-r^6)*(21*x+2-t^4));
           6           4                      2  3
%0047: (- r  + u) (2 - t  + 21 x) (1 - x + y) 5  2
(%0048) factor(2*(a+u)*(-v+b)*(a*x+y)^2);
                                      2
%0048: -1 2 (a + u) (- b + v) (a x + y)
(%0049) factor(2*(a+u)*(-v+b)*(a*x+y)^2/((u^2-v^2)*(11*x+55)));
                                    2
       2 (a + u) (- b + v) (a x + y)
%0049: -----------------------------
        11 (- u + v) (u + v) (5 + x)
(%0050) factor((c^3*u+b*a)*(b*b*a+v*p^2*q^2*c));
               3        2      2  2
%0050: (a b + c  u) (a b  + c p  q  v)
(%0051) factor((2*z+y-x)*(y^3-a*x^2)*(b*z^2+y));
             2    3                          2
```

```
%0051: (- a x  + y ) (- x + y + 2 z) (y + b z )
(%0052) factor((a*a*b*z+d)*(2*a*b*b*z+c));
            2               2
%0052: (d + a  b z) (c + 2 a b  z)
(%0053) factor((a*a*b*z+d)*(2*a*b*b*z+c)*((u+a)*x+1));
                              2               2
%0053: (1 + (a + u) x) (d + a  b z) (c + 2 a b  z)
(%0054) factor(a*(z+1)/4);
       a (1 + z)
%0054: ---------
           2
          2
```

The rest of this section documents commands from the factoring package. To use this package, execute the following command from the JACAL prompt:

```
require("ff");
```

Several of these commands return a matrix. The first column contains the factors and the second column contains the corresponding exponent.

`sff` *poly* [Command]

Given a primitive univariate polynomial *poly*, calculate the square free factorisation of *poly*. A *primitive* polynomial is one with no factors (other than units) common to all its coefficients.

`ffsff` *poly p* [Command]
`ffsff` *poly p m* [Command]

Given a monic polynomial *poly*, a prime *p*, and a positive integer *m*, calculate the square free factorisation of *poly* in GF(p^m)[x]. If *m* is not supplied, `1` is assumed.

```
(%0059) ffsff(x^5+x^3+1, 53);
       [                 2    3    ]
       [ 16 - 22 x + 26 x  + x   1 ]
%0059: [                           ]
       [        -13 + x          2 ]
```

`berl` *poly n* [Command]

Given a square-free univariate polynomial *poly* and an integer power of a prime, *q*, returns (as a bunch) the irreducible factors of *poly*.

```
(%0060) berl(x^5+x^3+2, 53);
                             2               2
%0060: [1 + x, 5 - 26 x + x , 11 + 25 x + x ]
```

`parfrac` *polyratio* [Command]

Returns the partial fraction expansion of a rational univariate polynomial *polyratio*. The denominator of *polyratio* must be square free. This code is still being developed.

4 Transcendental Functions

JACAL has the ability to manipulate and simplify functions defined by first order linear differential equations.

The following functions and their reductions are defined by statements in `init.math`.

4.1 Exponentials and Logarithms

`exp` *expression* [Operator]

Represents the exponential function of scalar *expression*.

```
(%0043) exp(1+x);
%0043: exp(1 + x)
(%0044) exp(0);
%0044: 1
```

`log` *expression* [Operator]

Represents the natural logarithm of scalar *expression*.

```
(%0045) log(1+x);
%0045: log(1 + x)
(%0046) exp(5*log(a));
        5
%0046: a
```

`%expt` *expression exponent* [Operator]

`%expt` is equivalent to *`exponent`*`*log(`*`expression`*`)`. If *exponent* is an integer, it simplifies to a power of *expression*; a fractional *exponent* does not.

```
(%0048) %expt(a,5);
        5
%0048: a
(%0049) %expt(a,1/5);
           log(a)
%0049: exp(------)
             5
(%0050) a^(1/5);
        1/5
%0050: a
```

4.2 Trigonometric Functions

`tan` *expression* [Operator]
`atan` *expression* [Operator]

These are the tangent and arctangent functions, from which all the other trigonometric functions are defined.

`cot` *expression* [Operator]
`acot` *expression* [Operator]

`sin` *expression* [Operator]
`asin` *expression* [Operator]
`cos` *expression* [Operator]
`acos` *expression* [Operator]
`sec` *expression* [Operator]
`asec` *expression* [Operator]
`csc` *expression* [Operator]
`acsc` *expression* [Operator]

The standard trigonometric functions and their inverses.

```
(%0051) tan(atan(x));
%0051: x
(%0052) atan(tan(x));
%0052: x
(%0053) tan(2*atan(x));
        2 x
%0053: ------
             2
       1 - x
(%0054) tan(3*atan(x));
                3
       - 3 x + x
%0054: ----------
                2
       -1 + 3 x
(%0055) tan(4*atan(x));
                   3
        - 4 x + 4 x
%0055: --------------
                2    4
       -1 + 6 x  - x
(%0056) atan(2*tan(x));
%0056: atan(2 tan(x))
(%0057) sin(3*asin(x));
                    3
%0057: - 3 x + 4 x
```

4.3 Lambert W function

`%W` *expression* [Command]

The Lambert W function defined as the inverse of `z=%W*exp(%W)`.

`lambert_w` *expression* [Command]

This equivalent function has the name which MAXIMA uses. It is defined as:

```
lambert_w(z) :: lambert_w(z)'/z'=lambert_w(z)/(z*(1+lambert_w(z)));
```

`lambert_w(log(x)*exp(log(x)))` does not reduce to `log(x)` because the principal branch of the Lambert-W function reduces only when `x>1/e`.

4.4 Defining Differential Equations

These are the commands used to define transcendental functions.

`::` [Command]

func(x) :: func(x)'/x' = expression

Where *expression* is a function of *func*(x) and/or x, defines the differential equation for *func*.

`::~` [Command]

inv ::~ func;
inv(x) ::~ expression;

Defines *inv* as the inverse of *func* or *expression*.

`::=` [Command]

func(const1) ::= const2

Where *const1* and *const2* evaluate to constants, sets an initial condition for the differential equation defining *func*.

The order in which initial conditions are defined matters; the first initial condition defined is used to vet potential solutions for differential equations.

The file `init.math` also defines expression-valued integer recurrences which generate the reduction of a transcendental function composed with an integer multiple of its inverse function. In the future these recurrences will be generated automatically from the differential equations.

`:=` [Command]
`::=` [Command]

ident[index] := scalar-expression

Defines *ident* as a memoized recurrence. The *scalar-expression* can be any expression, and may include references to *ident* or other memoized recurrences with integer argument `[`*ixpr*`]` where *ixpr* evaluates to a non-negative integer. The result of such a reference may be a function; that function can take arguments within parentheses after a `[`*ixpr*`]` argument.

ident[integer] ::= scalar-expression

Sets an initial condition. This is necessary to prevent runaway recursion if a recurrence does index arithmetic. Factorial can be defined as:

```
(%0039) fact[n] := n*fact[n-1];
fact[n]: n * fact[n - 1]
(%0039) fact[0] ::= 1;
(%0040) fact[3];
%0040: 6
(%0041) fact[7];
%0041: 5040
```

5 Calculus

5.1 Differential Operator

`differential` *expr* [Operator]

`'` *expr* [Operator]

The Jacal command `differential` computes the derivative of the expression *expr* with respect to a generic derivation. It is generic in the sense that nothing is assumed about its effect on the individual variables. The derivation is denoted by a right quote.

```
(%0061) differential(x^2+y^3);
                     2
%0061: 2 x x' + 3 y  y'
(%0062) (x^2+y^3)';
                     2
%0062: 2 x x' + 3 y  y'
```

5.2 Derivatives

`diff` *expr var1 ...* [Command]

The Jacal command `diff` computes the derivative of the expression *expr* with respect to *var1*, ...

```
(%0063) diff(x^2+y^3,y);
            2
%0063: 3 y
```

`partial` *expr var1 ...* [Command]

The Jacal command `partial` computes the partial derivative of the expression *expr* with respect to *var1*, ...

```
(%0064) partial(x^2+@1^3,1);
             2
%0064: 3 @1
```

5.3 Integration

`integrate` *expr var* [Command]

Returns the indefinite integral of rational expression *expr*, if that integral is a rational expression containing at most one radical involving *var*.

```
(%0065) integrate((3+x^2)*(1+x^2)^(2/3)/(3+6*x^2+3*x^4),x);
                2 2/3
       x (1 + x )
%0065: ------------
                2
          1 + x
(%0066) integrate((1+x^2)^(2/3),x);

;;; could-not-find-algebraic-anti-derivative
 non-decreasing-rxd 2 vs 0
(%0066) integrate(x*(1+x^2)^(2/3),x);
                2        2 2/3
       (3 + 3 x ) (1 + x )
%0066: ----------------------
                 10
```

`integrate` *expr var a b* [Command]

If the indefinite integral of rational expression *expr* is a rational expression (optionally including a radical involving *var*), then `integrate` returns the difference of that integral evaluated at *b* and *a*.

```
(%0067) integrate(x*(1+x^2)^(2/3),x,0,1);
                2/3
       -3 + 6 2
%0067: -----------
           10
```

6 Matrices and Tensors

In JACAL, a matrix is just a `bunch` of equal length `bunch`s, and this is the structure that the matrix operations currently supported by JACAL (ncmult(), `^^`, transpose(), etc.) expect. A row-vector is coded like `[[a,b,c]]`; a column-vector is coded by `[[a],[b],[c]]` or `[[a,b,c]]^^t` or `[a,b,c]^^t`.

6.1 Generating Matrices

`bunch` *elt_1 elt_2 . . .* [Operator]
`[`*elt_1*`, `*elt_2*`, ...]`

To collect any number of Jacal objects into a bunch, simply enclose them in square brackets. For example, to make the bunch whose elements are `1`, `2`, `4`, type `[1, 2, 4]`. One can also nest bunches, for example, `[1, [[1, 3], [2, 5]], [1, 4]]`.

A bunch whose only element is `[1, 2, 3]` is a row vector, which displays as `[1 2 3]`. The bunch has commas; the row vector does not.

```
(%0068) B1:bunch(1, 2, 3);
B1: [1, 2, 3]
(%0068) B2:[B1];
B2: [ 1  2  3 ]
(%0068) [B2];
%0068: [ [1, 2, 3] ]
(%0069) [[[1, 2, 3]]];
%0069: [ [1, 2, 3] ]
```

`flatten` *bnch* [Operator]

Removes bunch nesting from *bnch*, returning a single bunch of the constituent expressions and equations.

```
(%0070) flatten([a, [b, [c, d]], [5]]);
%0070: [a, b, c, d, 5]
```

`ident` *n* [Command]

The command `ident` takes as argument a positive integer *n* and returns an *n*x*n* identity matrix. This is sometimes more convenient than obtaining this same matrix using the command `scalarmatrix`.

```
(%0071) ident(4);
        [ 1  0  0  0 ]
        [             ]
        [ 0  1  0  0 ]
%0071:  [             ]
        [ 0  0  1  0 ]
        [             ]
        [ 0  0  0  1 ]
```

`scalarmatrix` *size entry* [Command]

The command `scalarmatrix` takes as inputs a positive integer *size* and an algebraic expression *entry* and returns an `n * n` diagonal matrix whose diagonal entries are all equal to *entry*, where `n = size`.

```
(%0072) scalarmatrix(3, 6);
       [ 6  0  0 ]
       [         ]
%0072: [ 0  6  0 ]
       [         ]
       [ 0  0  6 ]
```

`diagmatrix` *list* [Command]

The Jacal command `diagmatrix` takes as input a list of objects and returns the diagonal matrix having those objects as diagonal entries. In case one wants all of the diagonal entries to be equal, it is more convenient to use the command `scalarmatrix`.

```
(%0073) diagmatrix(12,3,a,s^2);
       [ 12  0  0  0  ]
       [              ]
       [ 0   3  0  0  ]
%0073: [              ]
       [ 0   0  a  0  ]
       [              ]
       [ 0   0  0   2 ]
       [           s  ]
(%0074) diagmatrix([1,2],2);
       [ [1, 2]  0 ]
%0074: [           ]
       [   0     2 ]
```

`sylvester` *poly_1 poly_2 var* [Command]

Here, *poly_1* and *poly_2* are polynomials and *var* is a variable. The function `sylvester` returns the matrix introduced by Sylvester (*A Method of Determining By Mere Inspection the Derivatives from Two Equations of Any Degree, Phil.Mag.* 16 *(1840) pp. 132-135, Mathematical Papers, vol. I, pp. 54-57*) for computing the resultant of the two polynomials *poly_1* and *poly_2* with respect to the variable *var*. If one wants to compute the resultant itself, one can simply use the command `resultant` with the same syntax.

```
(%0075) sylvester(a0 + a1*x + a2*x^2 + a3*x^3, b0 + b1*x + b2*x^2, x);
       [ a3  a2  a1  a0  0  ]
       [                    ]
       [ 0   a3  a2  a1  a0 ]
       [                    ]
%0075: [ b2  b1  b0  0   0  ]
       [                    ]
       [ 0   b2  b1  b0  0  ]
       [                    ]
       [ 0   0   b2  b1  b0 ]
```

`genmatrix` *function rows cols* [Command]

The function `genmatrix` takes as arguments a *function* of two variables and two positive integers, *rows* and *cols*. It returns a matrix with the indicated numbers of rows and columns in which the (i,j)th entry is obtained by evaluating *function* at (i,j). The function may be defined in any of the ways available in Jacal, i.e previously by an explicit algebraic definition, by an explicit lambda expression or by an implicit lambda expression.

```
(%0076) genfun: @1^2+@2^2;
          2     2
genfun: @1  + @2
(%0076) genmatrix(genfun,3,5);
       [ 2   5   10  17  26 ]
       [                    ]
%0076: [ 5   8   13  20  29 ]
       [                    ]
       [ 10  13  18  25  34 ]
```

6.2 Matrix Parts

`rank` *matrix* [Command]

The rank of *matrix* is the maximal number of linearly independent columns of *matrix*, which is always equalt to the maximal number of linearly independent rows of *matrix*.

```
(%0077) rank([[0,0],[0,0]]);
%0077: 0
(%0078) rank([[0,0],[0,1]]);
%0078: 1
(%0079) rank([[2,0],[0,1]]);
%0079: 2
(%0080) rank([[b,c],[0,a]]);
%0080: 2
(%0081) rank([[b,c,d],[a,0,a],[e,f,a]]);
%0081: 3
```

`row` *matrix i* [Command]

The command `row` returns the *i*th row of the matrix *matrix*, where `i = int`. If *int* is larger than the number of rows of *matrix*, then Jacal prints an error message. The corresponding command for columns of a matrix is `col`.

```
(%0082) U1:[[1, 2, 3], [1, 5, 3]];
    [ 1  2  3 ]
U1: [         ]
    [ 1  5  3 ]
(%0082) row(U1, 2);
%0082: [1, 5, 3]
```

`col` *matrix integer* [Command]

The command `col` is used to extract a column of a matrix. Here, *matrix* is a matrix and *integer* is a positive integer. It is an error if that integer exceeds the number of columns.

```
(%0084) C1:[[1,2,4],[2,5,6]];
    [ 1  2  4 ]
C1: [         ]
    [ 2  5  6 ]
(%0084) col(C1,2);
       [ 2 ]
%0084: [   ]
       [ 5 ]
```

`minor` *matrix i j* [Command]

The command `minor` returns the submatrix of *matrix* obtained by deleting the *i*th row and the *j*th column.

```
(%0085) M3:[[1,2,3],[3,1,5],[5,2,7]];
    [ 1  2  3 ]
    [         ]
M3: [ 3  1  5 ]
    [         ]
    [ 5  2  7 ]
(%0085) minor(M3,3,1);
       [ 2  3 ]
%0085: [      ]
       [ 1  5 ]
```

`cofactor` *matrix i j* [Command]

The command `cofactor` returns the determinant of the *i*, *j* `minor` of *matrix*.

`rref` *bunch int_1 int_2 . . .* [Command]

The function `rref` is used to access elements of bunches. It can also access elements nested at lower levels in a bunch. In particular, it can also access matrix elements. In the above syntax, *bunch* is the bunch whose parts one wishes to access, and *n*, *int_1*, *int_2*, . . . , *int_n* are positive integers. It returns the *int_n*-th element of the *int_{n-1}*-th element of . . . of the *int_2*-th element of the *int_1*-th element of *bunch*. One can have `n = 0`. In that case, `rref` simply returns the bunch.

```
(%0088) [[1,2,3],[1,4,6],3] [2] [3];
%0088: 6
(%0089) [a,b] [2];
%0089: b
```

6.3 Matrix commands

`transpose` *matrix* [Command]

Computes the transpose of (*matrix*).

```
(%0000) transpose([[a,b],[c,d]]);
       [ a  c ]
%0000: [      ]
       [ b  d ]
(%0001) [[a,b],[c,d]]^^t;
       [ a  c ]
%0001: [      ]
       [ b  d ]
```

`determinant` *matrix* [Command]

The Jacal command `determinant` computes the determinant of a square matrix. Attempting to take the determinant of a non-square matrix will produce an error message.

```
(%0008) A1:[[1,2],[6,7]];
    [ 1  2 ]
A1: [      ]
    [ 6  7 ]
(%0008) determinant(A1);
%0008: -5
```

`charpoly` *matrix var* [Command]

The characteristic polynomial of *matrix*:

determinant(*matrix* - I *var*)

`.` *matrix1 matrix2* [Command]

Matrix multiplication.

```
(%0009) A1:[[1, 2, 3], [5, 2, 7]];
    [ 1  2  3 ]
A1: [         ]
    [ 5  2  7 ]
(%0009) A2:[[3, 2], [6, 4]];
    [ 3  2 ]
A2: [      ]
    [ 6  4 ]
(%0009) A2 . A1;
       [ 13  10  23 ]
%0009: [            ]
       [ 26  20  46 ]
```

`^^` *matrix exponent* [Command]

The infix operator `^^` is used for raising a square matrix to an integral power.

```
(%0010) A3:[[1, 0], [-1, 1]];
    [ 1   0 ]
A3: [       ]
    [ -1  1 ]
(%0010) A3^^3;
       [ 1   0 ]
%0010: [       ]
       [ -3  1 ]
```

Negative exponents raise the inverse matrix to a power.

```
(%0000) A4: [[a, b], [c, d]];
    [ a  b ]
A4: [      ]
    [ c  d ]
(%0000) A4^^-1;
       [      d            - b      ]
       [ -----------  ----------- ]
       [ - b c + a d  - b c + a d ]
       [                         ]
%0000: [     - c           a      ]
       [ -----------  ----------- ]
       [ - b c + a d  - b c + a d ]
(%0001) A4^^-2;
       [                2                    - a b - b d          ]
       [         b c + d                    ----------------------- ]
       [ ------------------------            2  2              2  2 ]
       [  2  2               2  2           b  c  - 2 a b c d + a  d ]
       [ b  c  - 2 a b c d + a  d                                    ]
       [                                        2                    ]
%0001: [        - a c - c d                    a  + b c              ]
       [ ------------------------  ----------------------- ]
       [  2  2               2  2   2  2              2  2 ]
       [ b  c  - 2 a b c d + a  d   b  c  - 2 a b c d + a  d ]

(%0002) A4 . A4.1;
       [ 1  0 ]
%0002: [      ]
       [ 0  1 ]
(%0003) A4.1 . A4;
       [ 1  0 ]
%0003: [      ]
       [ 0  1 ]
(%0004) A4.2 . A4 . A4;
       [ 1  0 ]
%0004: [      ]
       [ 0  1 ]
```

`dotproduct` *vector_1 vector_2* [Command]

The Jacal function `dotproduct` returns the dot product of two row vectors of the same length. It will also give the dot product of two matrices of the same size by computing the sum of the dot products of the corresponding rows or, what is the same, the trace of one matrix times the transpose of the other one.

```
(%0005) V1: [1,2,3]; V2: [3,1,5];
V1: [1, 2, 3]
V2: [3, 1, 5]
(%0005) dotproduct(V1, V2);
%0005: 20
```

`crossproduct` *vector_1 vector_2* [Command]

The Jacal command `crossproduct` computes the cross product of two vectors. By definition, the two vectors must each have three components.

```
(%0007) crossproduct([1,2,3],[4,2,5]);
%0007: [4, 7, -6]
```

6.4 Tensors

The `tensors` supported by JACAL are an extension of the matrix structure (i.e., a bunch of bunches of bunches . . .) with the added stipulation that all `dimensions` of the tensor be the same length (e.g., 4x4x4). The number of dimensions (indices) in a tensor is its rank: A scalar is a tensor of rank 0; a vector is a rank 1 tensor; a matrix has rank 2; and so on.

Further, just as matrix binary operations place restrictions on the matrices involved (e.g., the row/column length requirement for matrix multiplication), the tensor binary operations require that the dimensions of each tensor be of the same length. For example, you could not multiply a 3x3 tensor and a 4x4x4 tensor.

JACAL's tensors do not support the construct of contravariant and covariant indices. Users must keep track of this information themselves, and perform the necessary operations with an appropriate metric so that the "index gymnastics" is performed correctly.

JACAL currently supports four tensor operations: `tmult`, `contract`, `indexshift`, and `indexswap`. Each of these is described in detail below.

To use JACAL's tensor operations, execute the following command from the JACAL prompt:

require("tensor");

To see tensor examples, download `rw.math` from the JACAL website `https://www.gnu.org/software/jacal/`. Typing `batch("rw.math");` will execute the commands in the file, computing tensors for The Robertson-Walker Cosmology Model. You can then view a tensor by typing its label (name before ':') followed by a semicolon.

```
(%0001) metric_ll: diagmatrix((S(x4))^2/(1 - k * x1^2), (S(x4))^2 * x1^2, (S(x4))^2 *
(%0002) metric_uu: diagmatrix(1/metric_ll[1][1], 1/metric_ll[2][2], 1/metric_ll[3][3],
(%0002) metric_ul: ident(4)
(%0002) det_metric: determinant(metric_ll)
(%0002) d_metric_lll: indexshift([diff(metric_ll, x1), diff(metric_ll, x2), diff(metri
(%0002) Christoffel_ull: (tmult(metric_uu, d_metric_lll, 2, 1) + indexswap(tmult(metri
```

```
(%0002) d_Christoffel_ulll: indexshift([diff(Christoffel_ull, x1), diff(Christoffel_ul
(%0002) R_temp: d_Christoffel_ulll + indexshift(tmult(Christoffel_ull, Christoffel_ull
(%0002) Riemann_ulll: indexswap(R_temp, 3, 4) - R_temp
(%0002) Ricci_ll: contract(Riemann_ulll, 1, 3)
(%0002) Ricci_ul: tmult(metric_uu, Ricci_ll, 2, 1)
(%0002) Ricci_uu: tmult(Ricci_ul, metric_uu, 2, 1)
(%0002) scalar_curv: contract(Ricci_ul, 1, 2)
(%0002) Einstein_ll: Ricci_ll - scalar_curv * metric_ll/2
(%0002) Einstein_ul: Ricci_ul - scalar_curv * metric_ul/2
(%0002) Einstein_uu: Ricci_uu - scalar_curv * metric_uu/2
(%0002)
(%0002) Einstein_uu;
             [       2   2                                                  ]
             [ - k + k  x1         0                                        ]
             [ ------------                              0                  ]
             [          4          k                                        ]
             [   (S(x4))     --------------              0                0  ]
             [                    2        4                               ]
             [      0        - x1  (S(x4))                     2          0  ]
Einstein_uu: [                                    k + k (tan(x2))           ]
             [      0              0         -------------------------     0  ]
             [                                    2        4          2      ]
             [      0              0         - x1  (S(x4))  (tan(x2))    3 k  ]
             [                                                         -------- ]
             [                                         0                    2 ]
             [                                                        (S(x4))  ]
```

6.5 Tensor Multiplication

`tmult` *matrix_1 matrix_2 index_1 index_2* [Command]

`tmult` takes a minimum of two arguments which are the tensors on which the multiplication operation is to be performed.

With no additional arguments, `tmult` will produce the outer product of the two input tensors. The rank of the resulting tensor is the sum of the inputs' ranks, and the components of the result are formed from the pair-wise products of components of the inputs. For example, for the input tensors `x[a,b]` and `y[c]`

```
z:tmult(x,y); ⇒ z[a,b,c] = x[a,b]*y[c]
```

With an additional argument, `tmult` will produce the inner product of the two tensors on the specified index. For example, given `x[i,j]` and `y[k,l,m]`

```
z:tmult(x,y,3);
⇒
                          length
                          -----
                          \
                z[a,b,c] =  >   x[a,q] * y[b,c,q]
                          /
                          -----
                          q = 1
```

Note that in this case x only has 2 indices. All of JACAL's tensor operations modify index inputs to be between 1 and the rank of the tensor. Thus, in this example, the 3 is modified to 2 in the case of x. As another example, with `x[i,j,k]` and `y[l,m,n]`

```
z:tmult(x,y,2);
⇒
                            length
                            -----
                            \
                z[a,b,c,d] =  >   x[a,q,b] * y[c,q,d]
                            /
                            -----
                            q = 1
```

With four arguments, `tmult` produces an inner product of the two tensors on the specified indices. For example, for `x[i,j]` and `y[k,l,m]`

```
z:tmult(x,y,1,3);
⇒
                          length
                          -----
                          \
                z[a,b,c] =  >   x[q,a] * y[b,c,q]
                          /
                          -----
                          q = 1
```

Note that matrix multiplication is the special case of an inner product (of two "two dimensional matrices") on the second and first indices, respectively: `tmult(x,y,2,1)` ≡ `ncmult(x,y)`

Finally, tmult handles the case of a scalar times a tensor, in which case each component of the tensor is multiplied by the scalar.

6.6 Tensor contraction

`contract` *matrix index1 . . .* [Command]

The contraction operation produces a tensor of rank two less than a given tensor. It does this by performing a summation over two of the indices of the given tensor, as clarified in the examples below.

`contract` takes at least one argument which is the tensor on which the contraction operation is to be performed. One or two additional arguments may be provided to specify the indices to be used in the summation. If no additional arguments are provided, the summation is performed over the first and second indices. With one additional argument, the summation is over the specified index and the one following it (e.g., if 3 is specified, the third and fourth indices are used). With two additional arguments, the summation is performed over the indices specified. The actual indices used will be constrained to be between 1 and the rank of the tensor.

Examples:

1) For a square matrix (tensor of rank 2), `contract` returns a scalar that is the sum of the diagonal elements of the matrix.

2) Given `x[i,j,k,l]`, the command

```
y:contract(x,2,4);
```

produces:

```
                    length
                    -----
                    \
y[a,b] =             >    x[a,q,b,q]
                    /
                    -----
                    q = 1
```

Special cases: If `contract` is given a scalar (rank 0 tensor) as input, it just returns the scalar. For a vector (tensor of rank 1), `contract` returns a scalar that is the sum of the elements of the vector.

6.7 Shifting of Tensor Indices

`indexshift` *matrix index1 . . .* [Command]

`indexshift` rearranges the indices of a tensor. It is one of two generalizations of the matrix transpose operation (cf. `indexswap`).

`indexshift` takes at least one argument which is the tensor on which the index shifting is to be performed. One or two additional arguments may be provided to specify the index and the position to which it is to be shifted. If no additional arguments are provided, the first index of the tensor is shifted to the second position (equivalent the matrix transpose operation). If one additional argument is provided, it specifies the index to be shifted, and that index will be shifted "to the right" one position (e.g., if 3 is specified, the third index will be shifted to the forth position). If two additional arguments are provided, the first specifies the index and the second specifies the position to which it is to be shifted. The actual index shifted and its shifted position will be constrained to be between 1 and the rank of the tensor.

For example, given `x[a,b,c,d]`, the command `y:indexshift(x,1,3);` produces a tensor y such that `y[a,b,c,d]` $\equiv$ `x[b,c,a,d]`. In this example, the element that was in position `[a,b,c,d]` in x will be in position `[b,c,a,d]` in y.

Special cases: If `indexshift` is given a scalar (rank 0 tensor) as input, it just returns the scalar. For a vector (tensor of rank 1), `indexshift` transposes the 1-by-n matrix (row vector) to an n-by-1 matrix (column vector).

6.8 Swapping of Tensor Indices

`indexswap` *tensor . . .* [Command]

`indexswap` rearranges the indices of a tensor. It is one of two generalizations of the matrix transpose operation (cf. `indexshift`).

`indexswap` takes at least one argument which is the tensor on which index swapping is to be performed. One or two additional arguments may be provided to specify the indices to be swapped. If no additional arguments are provided, the first and second indices of the tensor are swapped (equivalent the matrix transpose operation). With one additional argument, the specified index is swapped with the one following it (e.g., if 2 is specified, the second and third indices will be swapped). If two additional arguments are provided, they specify the indices to be swapped. The actual indices used will be constrained to be between 1 and the rank of the tensor.

For example, given x[a,b,c,d], the command `y:indexswap(x,2,4);` produces a tensor y such that y[a,b,c,d] = x[a,d,c,b]. In this example, the element that was in position [a,b,c,d] in x will be in position [a,d,c,b] in y.

Special cases: If `indexswap` is given a scalar (rank 0 tensor) as input, it just returns the scalar. For a vector (tensor of rank 1), `indexswap` transposes the 1-by-n matrix (row vector) to an n-by-1 matrix (column vector).

7 Lambda Calculus

`lambda` *varlist expression* [Operator]

Jacal has the ability to work with lambda expressions, via the command `lambda`. Furthermore, Jacal always converts user definitions of functions by any method into lambda expressions and converts the dummy variables of the function definition into symbols such as *1*, *2*, ... Jacal can manipulate lambda expressions by manipulating their function parts, as in '`e14`' below. Jacal can also invert a function using the command `finv`.

```
(%0012) lambda([x],x^2);
          2
%0012: @1
(%0013) lambda([x,y,z],x*y*z);
%0013: @1 @2 @3
(%0014) %0012 + %0013;
          2
%0014: @1  + @1 @2 @3
```

`elementwise` *function matrix1 matrix2 ...* [Command]

The arguments *matrix1*, *matrix2*, ... must have the same shape. The command `elementwise` returns a new matrix formed by applying *function* to each tuple of elements of *matrix1*, *matrix2*, ...

```
(%0015) elementwise(foo,[a, b], [c, d]);
%0015: [foo(a, c), foo(b, d)]
(%0016) elementwise(@1+5*@2,[a, b], [c, d]);
%0016: [a + 5 c, b + 5 d]
(%0017) elementwise(@1-@2,[9,8,7],[[1,0],[4,5],[6,3]]);
        [ 8  9 ]
        [      ]
%0017:  [ 4  3 ]
        [      ]
        [ 1  4 ]
```

`finv` *function* [Command]

function`^^-1`

The command `finv` takes as input a function of one variable and returns the inverse of that function. The function may be defined in any of the ways permitted in Jacal, i.e. by an explicit algebraic definition, by an explicit lambda expression or by an implicit lamba expression. If *f* is the function, then typing `f^^-1` has the same effect as typing `finv(`*f*`)`.

```
(%0008) w(t):=t+1;
w(t): t + 1
(%0009) finv(w);
%0009: -1 + @1
```

8 Miscellaneous

`%` [Command]

The symbol `%` represents the last expression obtained by Jacal. It can be used in formulas like any other constant or variable or expression.

```
(%0010) 5+x;
%0010: 5 + x
(%0011) %^2-%;
                      2
%0011: 20 + 9 x + x
```

`batch` *filename* [Command]

The command `batch` is used to read in a file containing scripts of Jacal commands. Here, *filename* is a string in double quotes. The precise way in which one refers to a file is, of course, system dependent.

`tex` *expr* [Command]
`scheme` *expr* [Command]
`disp2d` *expr* [Command]
`standard` *expr* [Command]
`std` *expr* [Command]

Displays *expr* in TeX, the Scheme programming language, Jacal's two-dimensional output format, Jacal's infix input format, or compressed infix format respectively.

If one these top-level function calls is prefixed with `%horner`, the output will be in Horner's rule format, multiplication instead of exponentiation.

```
(%0002) disp2d((x-1)^3/(x+1)^3);
              2    3
-1 + 3 x - 3 x  + x
--------------------
             2    3
1 + 3 x + 3 x  + x
(%0002) horner(disp2d((x-1)^3/(x+1)^3));
-1 + (3 + (-3 + x) x) x
-----------------------
 1 + (3 + (3 + x) x) x
(%0002) scheme((x-1)^3/(x+1)^3);
(/ (+ (- (+ -1 (* 3 x)) (* 3 (expt x 2)))
      (expt x 3))
   (+ 1 (* 3 x) (* 3 (expt x 2)) (expt x 3)))
(%0002) standard((x-1)^3/(x+1)^3);
(-1 + 3 * x - 3 * x^2 + x^3)/(1 + 3 * x + 3 * x^2 + x^3)
(%0002) std((x-1)^3/(x+1)^3);
(-1+3*x-3*x^2+x^3)/(1+3*x+3*x^2+x^3)
(%0002) tex((x-1)^3/(x+1)^3);
{{-1+3\,x-3\,x^{2}+x^{3}}\over{1+3\,x+3\,x^{2}+x^{3}}}
```

`tex "`*string*`"` [Command]
Read TeX expression *string*. The `tex` command reads its double-quoted argument as a TeX expression

`scheme "`*string*`"` [Command]
Read Scheme double-quoted *string* argument as Scheme cdoe.

```
(%0000) scheme("(- (expt b 2) (* 4 a c))");
        2
%0000: b  - 4 a c
```

`disp2d "`*string*`"` [Command]
`standard "`*string*`"` [Command]
`std "`*string*`"` [Command]
Reads double-quoted *string* in Jacal's infix input format.

`commands` [Command]
The command `commands` produces a list of all of the command available in Jacal. It is called as a function of no arguments.

commands();
```
% * + - / = ^ ^^ abs args arity augcoefmatrix b+/- b-/+ batch bunch cabs
canon chain chainables charpoly coeff coeffs coefmatrix cofactor col
commands compose content continue crossproduct decompose degree denom
depends derivative describe determinant diagmatrix diff
differential discriminant disp2d divide dotproduct elementwise eliminate
equatecoeffs example extensions definition factorial factors finv flatten
func gcd genmatrix help ident imagpart integrate interp interp.lagrange
interp.neville interp.newton jacobi jacobian listofvars load matrix minor
mod monomial ncmult negate normalize num or over parallel partial poly
polyelim prime? qed quit rank raw realpart require restart resultant row
rref scalarmatrix scheme scheme2d set shade shadows show
squarefreefactors squarefreefactorslist standard std sylvester system
taylor terms tex transcript transpose u+/- u-/+ varpri vd verify wronski
wronskian
```

`describe` *command* [Command]
The command `describe` is the heart of the builtin help facility of Jacal. Here, *command* is a string which is the name of a command and `describe` produces a brief description of the command and in many cases includes an example of its use. Together with the command `commands()`, which prints a list of all available Jacal commands, and the command `example`, which gives an example of the use of the command, one can in principle use Jacal without a manual after one has learned how to get started.

```
(%0012) describe(col);
built-in-operation
column.  column of a matrix
(%0012) describe(resultant);
built-in-operation
resultant.  The result of eliminating a variable between 2
equations (or polynomials).
(%0012) describe(/);
built-in-operation
Quotient, division, divide, over.
a/b
```

`example` *command* [Command]

Here, *command* is a string which is the name of a Jacal command. `example` gives an example of the use of the command. See also Chapter 8 [Miscellaneous], page 47.

```
(%0013) example(*);
a * 7
%0013: 7 a
```

Note: the command `example` executes the example it gives. This can lead to unpredictable results if the variables and constants in the example have already been given values by the user.

`load` *string* [Command]

The Jacal command `load` takes as input a string and reads in a '`Scheme`' file whose name is obtained by appending the extension `.scm` to the string. If you want to read in a file of Jacal commands, do not use `load`. Instead use the command `batch`. To load in the file `tensor.scm`,

```
(%0014) load("tensor.scm");
"tensor.scm"
```

`qed` [Command]

Exit from Jacal to Scheme. With interactive Scheme systems (such as SCM), It does not return you to the operating system. Instead it suspends Jacal and returns you to the underlying scheme. You can return to the Jacal session where you left off by simply typing `(math)`. If you do not wish to return to Jacal but really want to terminate the session and return to the operating system, then after typing `qed;`, type `(slib:exit)` or use `quit`.

`quit` [Command]

Exit directly from Jacal to the operating system. You will not be able to continue your Jacal session.

```
(%0015) qed;
scheme
> (math)
Type qed; to return to Scheme, type help; for help.
(%0015) quit();
unix>
```

`system` *command* [Command]

One can issue commands to the operating system without leaving Jacal. To do this, one uses the command `system`. For example, in a UNIX operating system, the command `system("ls");` will print the directory. One way in which the command `system` might be especially useful is to edit files containing Jacal scripts without leaving Jacal, particularly in non-UNIX machines or on machines without GNU emacs.

```
(%0000) system("echo hi there");
hi there
%0000: 0
```

`terms` [Command]

Prints a copy of the GNU General Public License

```
(%0001) terms();
                    GNU GENERAL PUBLIC LICENSE
                       Version 3, 29 June 2007

 Copyright (C) 2007 Free Software Foundation, Inc. <http://fsf.org/>
 Everyone is permitted to copy and distribute verbatim copies
 of this license document, but changing it is not allowed.
```

[rest deleted for brevity]

`transcript` *string* [Command]

The command `transcript` allows one to record a Jacal session. It is called with the syntax `transcript(`*string*`);`, where *string* is the name of the file in which one wants to keep the transcript of the session. When one wishes to stop recording, one types `transcript();`. One is then free to use `transcript` again later in the session on another file. One can use it on the same file, but the file is overwritten. Presently, the command `transcript` does not echo commands to a file.

```
(%0001) a:[1,2,3];
a: [1, 2, 3]
(%0001) transcript("foo");
"foo"
(%0002) a;
a: [1, 2, 3]
(%0002) transcript();
(%0002) system("cat foo");


(%0002) a;

(%0002) transcript();%0002: 0
```

`set` *flag value* [Command]

There are various flags that the Jacal user can control, namely the Jacal command line prompt, the priority for printing terms in Jacal output, the input grammar and the output grammar. For a discussion of the various grammars please See Chapter 9

[Flags], page 52. The command `show` is closely related, allowing one to see what the current settings are.

`show` *flag* [Command]

The command `show` enables the Jacal user to examine the current setting of various flags as well as to list the flags that can be set by the user and to display other information. To change the settings of the flags, use the command `set`. To see all the information accessible through the `show` command, type `show all`. To see the available grammars, type `show grammars`. To see the current input grammar type `show ingrammar`. To see the current output grammar, type `show outgrammar`. To see the current variable ordering, type `show ordering`.

```
(%0003) show all;

     all debug echogrammar elims grammars horner ingrammar linkradicals
     ordering outgrammar page phases priority prompt trace version width

(%0003) show prompt;
%0003: %0003

(%0000) show ordering;

     @ %W (exp(@2*log(@1)))' ((tanh(@1))^(1/2))' ((1+(tan(@1))^2)^(1/2))'
     ((-1+(tanh(@1))^2)^(1/2))' (tanh(@1))' (tan(@1))' (log(@1))' (exp(@1^2))'
     (exp(@1))' (exp(1))' (exp(-@2^2))' (exp(-@1^2))' (erf(@1*%i))' (erf(@1))'
     (atanh(1/@1))' (atan(1/@1))' (atan(1))'
     (atan((-@1*%i*(-1+@1^2)^(1/2))/(-1+@1^2)))'
     (atan((-1+@1^2)^(1/2)/(-1+@1^2)))' (atan((-1+@1^2)^(1/2)))'
     (atan(%i*(-1+@1^2)^(1/2)/@1))' exp(@2*log(@1)) (tanh(@1))^(1/2)
     (1+(tan(@1))^2)^(1/2) (-1+(tanh(@1))^2)^(1/2) tanh(@1) tan(@1) log(@1)
     exp(@1^2) exp(@1) exp(1) exp(-@2^2) exp(-@1^2) erf(@1*%i) erf(@1)
     atanh(1/@1) atan(1/@1) atan(1) atan((-@1*%i*(-1+@1^2)^(1/2))/(-1+@1^2))
     atan((-1+@1^2)^(1/2)/(-1+@1^2)) atan((-1+@1^2)^(1/2))
     atan(%i*(-1+@1^2)^(1/2)/@1) tanh tan log fre exp erf atanh atan lambert_w
     ((atan(1))^(1/2))' (atan(1))^(1/2) @2' @1' (@1^(1/3))' (@1^(1/2))'
     ((-1+@1^2)^(1/2))' %i' @' %tanhQ[@1] %tanhQ[-1+@1] %tanhP[@1]
     %tanhP[-1+@1] %tanQ[@1] %tanQ[-1+@1] %tanP[@1] %tanP[-1+@1] %expPQ[-1+@1]
     @3 @2: @2 @1^(1/3) @1^(1/2) (-1+@1^2)^(1/2) %i @1: @1 %sqrt1 y x t
     ordering c b a ? ::@ %tanhQ %tanhPQ %tanhP %tanQ %tanPQ %tanP %expPQ
     "Currently in math mode."
(%0000) show outgrammar;
%0000: disp2d
(%0001) show ingrammar;
%0001: standard
(%0002) show grammars;
%0002: [null, raw, scheme, scheme2d, std, standard, disp2d, tex]
```

9 Flags

`prompt` *string* [Flag]

If one changes the prompt, *string* is a string of alphanumeric characters without quotes. After this command is executed, subsequent commands will cause new prompts to be obtained from *string* by incrementing it. If the prompt ends in a letter, it will be treated as a digit in base 26 and incremented. If it ends in a string of digits, that string will be treated as a number in base 10 and incremented. The remaining characters in the string will play no role in this incrementation.

```
(%0003) set prompt "az9Z";
(az9Z) a+b;
az9Z: a + b
(az9AA) a+b;
az9AA: a + b
(az9AB) set prompt "ok99";
(ok99) a+b;
ok99: a + b
(ok100) b+c;
ok100: b + c
```

`ingrammar` *grammar* [Flag]

`outgrammar` *grammar* [Flag]

The following examples show how one changes the input grammar or the output grammar.

```
(%0000) a:[[[1,2,3]]];
a: [ [1, 2, 3] ]
(%0000) set outgrammar standard;
(%0000) a;
a: [[[1, 2, 3]]]
(%0000) set outgrammar scheme;
(%0000) a;
(define a #(#(#(1 2 3))))

(%0000) (1+x)^5;
(define %0000 (+ 1 (* 5 x) (* 10 (expt x 2)) (* 10 (expt x 3)) (* 5 (expt x 4)) (

(%0001) set ingrammar scheme;
(%0001) (+ %0000 1);
(define %0002 (+ 2 (* 5 x) (* 10 (expt x 2)) (* 10 (expt x 3)) (* 5 (expt x 4)) (

(set ingrammar disp2d)
(%0003) diagmatrix(3,6);
(define %0003 #(#(3 0) #(0 6)))

(%0004) set outgrammar disp2d;
(%0004) %0003;
       [ 3  0 ]
%0003: [      ]
       [ 0  6 ]
(%0004) set outgrammar standard;
(%0004) %0003;
%0003: [[3, 0], [0, 6]]
```

Note that in the above examples, it is possible to input and output expressions in scheme by setting the ingrammar and/or outgrammar to `scheme`. Doing so result in linear output (as with `standard grammar`) as opposed to a two dimensional display (as with `disp2d`). The analogue of `disp2d` for scheme output is Scheme pretty-printing. To have such output, set the output grammar to `scheme2d`.

```
(%0004) set outgrammar scheme2d;
(%0004) (1+x)^5;
(define %0004
  (+ 1
     (* 5 x)
     (* 10 (expt x 2))
     (* 10 (expt x 3))
     (* 5 (expt x 4))
     (expt x 5)))
```

Jacal also allows for output to be automatically typeset in TEX. This can be quite useful if one wants to use the results of one's computations in published articles. Continuing with the example of `(1+x)^5` above, we have:

```
(%0005) set outgrammar tex;
(%0005) %0004;
%0004: 1+5\,x+10\,x^{2}+10\,x^{3}+5\,x^{4}+x^{5}
(%0005) (1+1/x)^3/(1-1/y)^4;
%0005: {{\left(1+3\,x+3\,x^{2}+x^{3}\right)\,y^{4}}\over{x^{3}-
  4\,x^{3}\,y+6\,x^{3}\,y^{2}-4\,x^{3}\,y^{3}+x^{3}\,y^{4}}}
```

After being included in a document in math display mode, these two examples will appear in the following way.

$$1+5\,x+10\,x^2+10\,x^3+5\,x^4+x^5$$

$$\frac{\left(1+3\,x+3\,x^2+x^3\right)\,y^4}{x^3-4\,x^3\,y+6\,x^3\,y^2-4\,x^3\,y^3+x^3\,y^4}$$

`priority` *int* [Flag]

The following examples show how to set the priority of printing terms.

```
(%0000) a: [[[1,2,3]]];
a: [ [1, 2, 3] ]
(%0000) show priority a;

;;; expl->var not a simple variable:  [[[1 2 3]]]

(%0000) show priority b;
%0000: 3000
(%0001) show priority c;
%0001: 3000
(%0002) b+c;
%0002: b + c
(%0003) c+b;
%0003: b + c
(%0004) set priority b 2000;
(%0004) b+c;
%0004: b + c
(%0005) set priority b 4000;
(%0005) b+c;
%0005: c + b
(%0006) %0002;
%0002: b + c
```

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